CS 360 — Assignment 2 Solutions

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1. The language $L = \{a^k bw \mid w \in \{a,b\}^k\}$ is not regular. We will show this via the pumping lemma. Suppose L is regular and let $n > 0$ be the pumping length for L. Let $u = a^n ba^n$. We factor $u = xyz$ by $x = a^s$ for $s \geq 0$, $y = a^t$ for $t > 0$ and $s + t \leq n$, and $z = a^{n-s-t}ba^n$. Then to satisfy the pumping lemma, we must have $xy^iz \in L$ for all $i \in \mathbb{N}$. Observe that $xy^iz = a^{s+i\cdot t+n-s-t}ba^n$. Then $xy^iz \in L$ only when $i = 1$ since the suffix following b in xy^iz must have length exactly $n + (i-1)t$. Thus, L does not satisfy the pumping lemma and L is not regular as assumed.

2. The language $L = \{a^ib^j \mid i + j = 0 \bmod 4\}$ is regular. There are multiple ways to show this; here, we'll show that L is matched by the following regular expression:

$$(a^4)^*(b^4)^* + a(a^4)^*b^3(b^4)^* + a^2(a^4)^*b^2(b^4)^* + a^3(a^4)^*b(b^4)^*.$$

   In each term, the expression $(a^4)^*$ corresponds to the language of words consisting of $4n$ a's (and the same for b's). Thus, the first term of the expression handles the case when both i and j are divisible by 4.

   For the other terms, we modify the expression slightly by concatenating it with a word that changes the number of a's that are matched. The second term handles the case when $i = 1 \bmod 4$ and $j = 3 \bmod 4$ by adding a and b^3. The third term handles the case when both i and j are $2 \bmod 4$ by concatenating a^2 and b^2. The fourth term matches those words where $i = 3 \bmod 4$ and $j = 1 \bmod 4$ by concatenating a^3 and b. In every case, $i + j = 0 \bmod 4$ as required.

   Taking the union of the four terms results in L. Each term clearly matches a regular language and there are finitely many terms, so taking the union of all of them also results in a regular language.

3. We will show L is not regular by contradiction. Suppose that L is regular. Then $\overline{L} = \{a^{k^2} \mid k \geq 0\}$ is also regular, by closure under complementation. Since $\overline{L}$ is regular, let n be the pumping length for $\overline{L}$. Let $w = a^{n^2}$ and we have $|w| \geq n$. We will factor $w = xyz$ and we have $y = a^m$ for some $1 \leq m \leq n$. Consider the word $xy^iz = a^{n^2+(i-1)k}$ for $i \in \mathbb{N}$. Choose $i = 2$ and we have $xy^2z = a^{n^2+k}$. However, since $1 \leq k \leq n$, we have

$$n^2 < n^2 + k < (n+1)^2 = n^2 + 2n + 1.$$

   Then $xy^2z \notin \overline{L}$, since $n^2 + k$ is not a perfect square. Thus, $\overline{L}$ is not regular as assumed. Furthermore, if $\overline{L}$ is not regular, then L is not regular as we had assumed.

4. The language $L = \{w \in \Sigma^* \mid w = \theta(w)\}$ is not regular. We will show this by the pumping lemma. Assume that L is regular and let $n > 0$ be the pumping length for L. We will choose the word A^nT^n. One can verify that $\theta(A^nT^n) = \theta(T^n)\theta(A^n) = A^nT^n$ and thus, $A^nT^n \in L$.

We factor A^nT^n into words xyz according to the pumping lemma, such that $x = A^s$ for $s \geq 0$, $y = A^t$ for $t > 0$ and $s + t \leq n$, and $z = A^{n-s-t}T^n$. Now we consider the word $A^sA^{it}A^{n-s-t}T^n$ and take $i = 0$. Then we have

$$\theta(A^{n-t}T^n) = A^nT^{n-t} \neq A^{n-t}T^n,$$

and thus, $xy^0z \notin L$. Therefore, L does not satisfy the pumping lemma and L is not regular as assumed.

5. (a) The grammar G generates the language $L(G) = \{a^nb^mc^n \mid m, n \geq 0\}$. To see this, we note that the grammar generates a word in two steps. First, on each application of the production $S \to aSc$, the grammar generates exactly one a on the left and one c on the right. Then, by applying the rule $S \to A$, we move to the next step. On A, the grammar can generate any number of b's in between the a's and c's already generated. Thus, we get a word of the form $a^nb^mc^n$.

   (b) The language $L(G)$, defined above, is not regular. Suppose for contradiction that $L(G)$ is regular. Then consider the language L' defined by

$$L' = L(G) \cap a^*c^* = \{a^nc^n \mid n \geq 0\}.$$

   Since $L(G)$ is regular and a^*c^* is regular, it must be the case that L' is regular, by closure of the regular languages under complementation. However, L' is known to be non-regular. Thus, $L(G)$ cannot be regular as assumed.

   This can also be shown with a pumping argument by considering the word a^nc^n, where n is the pumping length.