CS 360 — Assignment 1 Solutions

University of Waterloo, Spring 2018

1. We will prove this by induction on k. First, for our base case, let $k = 0$. Then,

$$u(vu)^0 = u = (uv)^0 u.$$

Now, let $k \geq 1$ and suppose that $u(vu)^\ell = (uv)^\ell u$ for all $\ell < k$. Then we have

$$\begin{aligned} u(vu)^k &= uvu(vu)^{k-1} && \text{since } k > 0 \\ &= uv(u(vu)^{k-1}) \\ &= uv(uv)^{k-1}u && \text{by induction hypothesis} \\ &= (uv)^k u. \end{aligned}$$

2. (a) The DFA A is defined $A = (\{q_0, q_1\}, \{0, 1\}, \delta, q_0, \{q_0\})$ where the transition function δ is specified by

$$\begin{aligned} \delta(q_0, 0) &= q_1 & \delta(q_0, 1) &= q_0 \\ \delta(q_1, 0) &= q_0 & \delta(q_1, 1) &= q_1 \end{aligned}$$

(b) We claim that the DFA A recognizes the language

$$L = \{w \in \{0, 1\}^* \mid |w|_0 = 0 \bmod 2\}.$$

That is, A recognizes the language of binary strings with an even number of 0s.

We will prove this by showing that a word w reaches state q_i if and only if $|w|_0 = i \bmod 2$. We will show this by induction on the length of w. First, for the base case, let $w = \varepsilon$. Then $\delta(q_0, w) = \delta(q_0, \varepsilon) = 0$ and $|\varepsilon|_0 = 0 = 0 \bmod 2$.

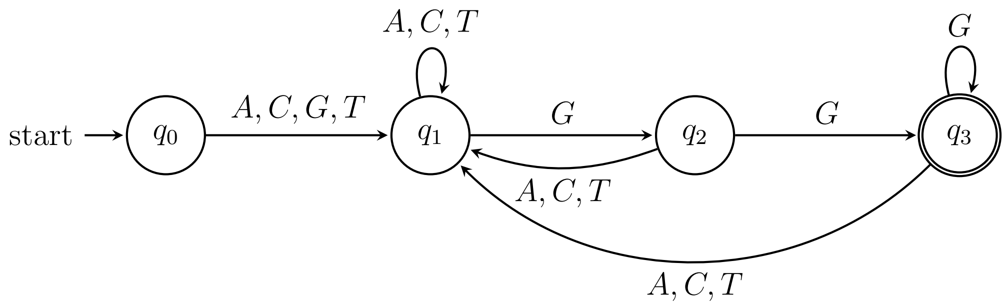
Now let $|w| > 0$ and assume that for every word w' with $|w'| < |w|$, $\delta(q_0, w') = q_i$ if and only if $|w'|_0 = i \bmod 2$. Let $w = w'a$ for a word $w' \in \{0, 1\}^*$ and $a \in \{0, 1\}$. Since $|w'| = k - 1$, by our inductive hypothesis, we have $\delta(q_0, w') = q_i$ if and only if $|w'|_0 = i \bmod 2$. There are two cases to consider.

i. If $i = 0$, then $|w'|_0 = 0 \bmod 2$. If $a = 0$, then $\delta(q_0, 0) = q_1$ and $|w|_0 = |w'|_0 + 1 = 1 \bmod 2$. Similarly, if $a = 1$, then $\delta(q_0, 0) = q_0)$ and $|w|_0 = |w'|_0 + 0 = 0 \bmod 2$.

ii. If $i = 1$, then $|w'|_0 = 1 \bmod 2$. If $a = 0$, then $\delta(q_1, 0) = q_0$ and $|w|_0 = |w'|_0 + 1 = 0 \bmod 2$. Similarly, if $a = 1$, then $\delta(q_1, 0) = q_1$ and $|w|_0 = |w'|_0 + 0 = 0 \bmod 2$.

Thus, we have shown that $\delta(q_0, w) = q_i$ if and only if $|w|_0 = i \bmod 2$. Since q_0 is the only accepting state of A, we have that A recognizes a word w if and only if $|w| = 0 \bmod 2$.

3. Let A be the DFA depicted in the state diagram that follows.



We will show that A recognizes L by showing that for a word $w \in \Sigma^*$, we have the following conditions:

(a) $\delta(q_0, w) = q_0$ iff $w = \varepsilon$,

(b) $\delta(q_0, w) = q_1$ iff $w = G$ or $w = w'a$ for some $w' \in \Sigma^*$ and $a \in \{A, C, T\}$,

(c) $\delta(q_0, w) = q_2$ iff $w = w'G$ for some $w' \in \Sigma^+$,

(d) $\delta(q_0, w) = q_3$ iff $w = w'GG$ for some $w' \in \Sigma^+$.

We will show this by induction on the length of w.

For our base case, we consider $|w| = 0$, which means $w = \varepsilon$. Since $w = \varepsilon$, we have $\delta(q_0, w) = q_0$ by definition. Thus the base case holds.

Now we consider $|w| > 0$ and assume that for all words $u \in \Sigma^*$ with $|u| < |w|$, u satisfies the conditions above. Let $w = w'a$ for $a \in \Sigma$ and $w' \in \Sigma^*$. We have the following cases to consider.

(a) If $a \in \{A, C, T\}$, then for every $w' \in \Sigma^*$, we have $\delta(\delta(q_0, w'), a) = q_1$, since for every state $q \in Q$, we have $\delta(q, a) = q_1$ by definition. Then this satisfies the condition that $\delta(q_0, w) = q_1$ if only if $w = w'a$ for $a \in \{A, C, T\}$ or $w = G$.

(b) If $a = G$, then we have the following cases to consider.

  i. If $\delta(q_0, w') = q_0$, then $w' = \varepsilon$ by our inductive hypothesis. Then we have $\delta(q_0, G) = q_1$ by definition and $w = G$, satisfying the condition that w is either G or ends in one of A, C, T.

  ii. If $\delta(q_0, w') = q_1$ then $w' = G$ or $w' = ub$ for some $u \in \Sigma^*$ and $b \in \{A, C, T\}$ by the induction hypothesis. We have $\delta(q_1, G) = q_2$ and $w = w'G$, as required.

  iii. If $\delta(q_0, w') = q_2$ then $w' = uG$ for some $u \in \Sigma^+$ by the induction hypothesis. We have $\delta(q_2, G) = q_3$ and $w = w'G = uGG$ for some $u \in \Sigma^+$ as required.

iv. If $\delta(q_0, w') = q_3$, then $w' = uGG$ for some $u \in \Sigma^+$ by the induction hypothesis. We have $\delta(q_3, G) = q_3$. Let $v = uG$ and we have $w = uGGG = vGG$, with $v \in \Sigma^+$ as required.

Since q_3 is the sole final state, A must only accept words of the form $w = w'GG$ for $w' \in \Sigma^+$. Since $w' \in \Sigma^+$, we can write $w' = ua$ for some $u \in \Sigma^*$ and $a \in \Sigma$ and we have $w = uaGG$. Thus, A recognizes all words ending in aGG with $a \in \Sigma$.

4. Since L is regular, it must be recognized by a DFA $A = (Q, \Sigma, \delta, q_0, F)$. We will show that L' is also regular by constructing a DFA B that recognizes it. Let $B = (Q, \Sigma, \delta, q_0, F')$, where Q, Σ, δ, q_0 are all as defined for A and the set of final states F' is defined by

$$F' = \{q \in Q \mid (\exists v \in \Sigma^*)\delta(q_0, v) \in F\}.$$

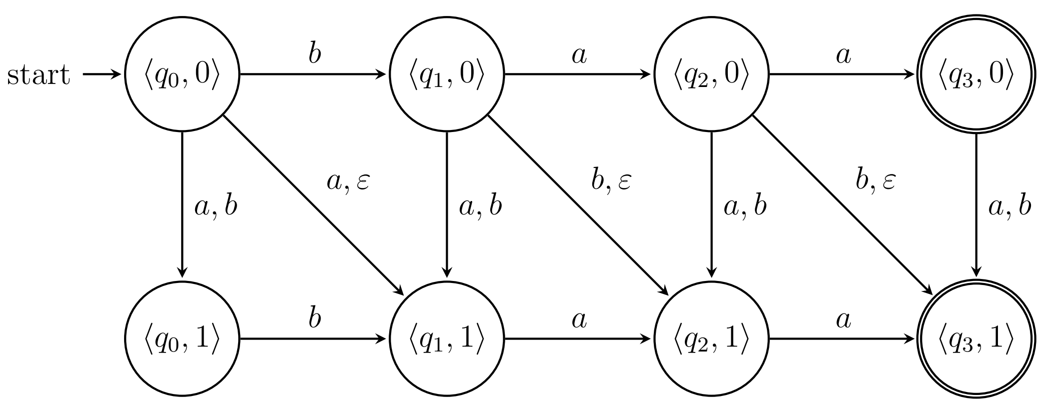
That is, F' consists of all states $q \in Q$ from which there exists a word that reaches a final state of A. Now, we will show that $L(B) = L'$.

First, we will show $L(B) \subseteq L'$. Suppose $w \in L(B)$. Let $q = \delta(q_0, w) \in F'$. By definition of F', there exists a word $v \in \Sigma^*$ such that $\delta(q, v) \in F$. But this means we have $\delta(\delta(q_0, w), v) = \delta(q_0, wv) \in F$ and therefore $wv \in L$. Thus, $w \in L'$ by definition of L'.

Next, we will show $L' \subseteq L(B)$. Suppose $w \in L'$. Then there exists a word $v \in \Sigma^*$ such that $wv \in L$. Thus, wv is recognized by A and we have $\delta(q_0, wv) \in F$. But this means there exists a state q such that $\delta(q_0, w) = q$ and $\delta(q, v) \in F$. This means we have $q \in F'$ by definition and thus $w \in L(B)$.

Thus, we have shown that $L(B) = L'$.

5. Let A be the ε-NFA depicted in the state diagram that follows.



We will show that A recognizes the language of words over $\{a.b\}$ with an edit distance of at most 1 from the word baa.

It is clear that travelling along each row of the NFA gives a path to accept the word baa. We claim that travelling from row 0 to row 1 corresponds to one edit operation. In particular, for $i = 0, 1, 2, 3$, going from $\langle q_i, 0\rangle$ to $\langle q_i, 1\rangle$ corresponds to an insertion of a symbol, while going from $\langle q_i, 0\rangle$ to $\langle q_{i+1}, 1\rangle$ corresponds to a deletion operation via the ε-transition and a substitution otherwise. Only one such edit operation can be performed; otherwise, the machine crashes, as there are no outgoing transitions corresponding to edit operations from states $\langle q_i, 1\rangle$.