CISC 462 — Computability and Complexity
Problem Set 2
Solutions

Queen's University, Winter 2017

1. (a) Rice's theorem cannot be used to show that A is decidable since the property involves the operation of the machine and not the language recognized by the machine.

   (b) Rice's theorem does not apply, since the property "$L(M)$ is recognizable" is trivial, since every Turing machine recognizes a recognizable language.

   (c) Rice's theorem can be used to show that C is undecidable, since it is a non-trivial semantic property.

   (d) Rice's theorem does not apply since $\overline{A_{TM}} \leq_M L(M)$ implies that $L(M)$ is unrecognizable. This is a trivial property since no Turing machine accepts an unrecognizable language.

2. We show that $FIN(\Sigma)$ has a correspondence with the set of binary words $\{0,1\}^*$, which we know to be countable. We also know that the set of words over Σ is countable and can be enumerated in lexicographic order $s_1, s_2, s_2, \ldots$. We define the characteristic sequence of a language $L \in FIN(\Sigma)$ to be a binary string $\mathbf{b} = b_1 b_2 \cdots b_n$ with

$$b_i = \begin{cases} 0 & \text{if } s_i \notin L, \\ 1 & \text{if } s_i \in L. \end{cases}$$

   If s_n is the lexicographically greatest string in L, then we define $s_j = \varepsilon$ for all $j > n$. The string s_n must exist since L is finite. Then every finite language L has a finite characteristic binary sequence and every finite binary string corresponds to a language over Σ. Thus, $FIN(\Sigma)$ is countable.

3. We define our problem as the following language

$$L = \{\langle M \rangle \mid \text{TM } M \text{ has a useless state}\}.$$

   Then we show that if we can decide L, then we can decide A_{TM}. Suppose there exists a Turing machine R that decides L. Then we can construct the following machine to decide A_{TM}:

   1. On input $\langle M, w \rangle$, construct the Turing machine M', which operates as follows:
      1. On input x, if $x \neq w$, then skip to the next step. Otherwise, simulate M on w.
      2. If M rejects w or $x \neq w$, then visit every state except q_A or q_R. We indicate that we are doing this by writing a special symbol ζ to the tape. After we have visited every state, enter q_R and *reject*.

3. If M accepts w, then *accept*.

2. Run R on $\langle M' \rangle$.
3. If R accepts, then *reject*; otherwise, *accept*.

If M does not accept w, then every state of M' is visited except for q_A. In this case, q_A is a useless state and R accepts. If M accepts w, then M' will enter the accepting state on input w and every other state is visited on input $x \neq w$. In this case R will reject. Thus, M' has a useless state iff $w \notin L(M)$.

Thus, if L is decidable, we can decide A_{TM}. Therefore L is undecidable.

4. If the alphabet is unary ($\Sigma = \{a\}$), then the strings only differ by length. Then the following algorithm decides Unary PCP:

   Given an instance of PCP $(u_1, v_1), \ldots, (u_k, v_k)$ over $\Sigma = \{a\}$,

   1. If there is a pair (u_i, v_i) with $u_i = v_i$, then this is a trivial match, so *accept*.
   2. If for every pair (u_i, v_i), we have $|u_i| > |v_i|$, then *reject*. If $|u_i| < |v_i|$ for all (u_i, v_i), then *reject*. In both cases, either the u_i's or v_i's will be larger and there will never be a match.
   3. Otherwise, there is a pair (u_i, v_i) with $|u_i| > |v_i|$ and a pair (u_j, v_j) with $|u_j| < |v_j$. Let $m = |u_i| - |v_i|$ and $n = |v_j| - |u_j|$. Then a solution is the sequence of $m + n$ integers
   $$i_1 = i_2 = \cdots = i_n = i, i_{n+1} = \cdots = i_{m+n} = j$$
   and thus we can *accept*.

5. (a) *False.* If $K \leq_M L$ is decidable and L is decidable, then K is decidable. However, A_{TM} is not decidable, so we cannot conclude anything about the decidability of B. (Note that this does not imply that B is undecidable; if B is decidable, there is a mapping reduction from B to A_{TM}.)

   (b) *True.* If $K \leq_M L$ and L is recognizable, then K is recognizable. Since A_{TM} is recognizable, B is recognizable.

6. Let M be a Turing machine and w be an input word such that $\langle M, w \rangle$ is encoded over an alphabet Σ and let ζ be a symbol not in Σ. Consider the language
   $$L = \{\langle M, w, \zeta^i \rangle \mid M \text{ accepts } w \text{ within } i \text{ steps}\}.$$
   L is decidable since it is guaranteed to halt. We can construct a machine that simulates M on w for up to i steps. Either M accepts w within i steps and accepts or M fails to accept within i steps and we halt the simulation after i steps and reject.

   Now, we define the following homomorphism φ by $\varphi(a) = a$ for all $a \in \Sigma$ and $\varphi(\zeta) = \varepsilon$, where ε is the empty string. Then $\varphi(L) = A_{TM}$. Since A_{TM} is undecidable, $\varphi(L)$ is undecidable and thus decidable languages are not closed under homomorphism.